PROJECTIVE SPECIAL LINEAR GROUPS $\mathrm{PSL}_4(q)$ ARE DETERMINED BY THE SET OF THEIR CHARACTER DEGREES

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ABSTRACT. Let G be a finite group and let $\mathrm{cd}(G)$ be the set of all irreducible complex character degrees of G. It was conjectured by Huppert in Illinois J. Math. 44 (2000) that, for every non-abelian finite simple group H, if $\mathrm{cd}(G) = \mathrm{cd}(H)$ then $G \cong H \times A$ for some abelian group A. In this paper, we confirm the conjecture for the family of projective special linear groups $\mathrm{PSL}_4(q)$ with $q \geq 13$.

1. Statement of the result

Let G be a finite group and let $\mathrm{cd}(G)$ denote the set of its irreducible complex character degrees. In general, the character degree set $\mathrm{cd}(G)$ does not completely determine the structure of G. It is possible for non-isomorphic groups to have the same set of character degrees. For example, the non-isomorphic groups D_8 and Q_8 not only have the same set of character degrees, but also share the same character table. The character degree set also cannot be used to distinguish between solvable and nilpotent groups. For example, if G is either Q_8 or S_3, then $\mathrm{cd}(G) = \{1, 2\}$. However, Huppert conjectured in the late 1990s that the nonabelian simple groups are essentially determined by the set of their character degrees. More explicitly, he proposed in [11] that

> *if G is a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same, then $G \cong H \times A$, where A is an abelian group.*

As abelian groups have only the trivial character degree and the character degrees of $H \times A$ are the products of the character degrees of H and those of A, this result is the best possible. The hypothesis that H is a nonabelian simple group is critical. There cannot be a corresponding result for solvable groups. For example, if we consider the solvable group Q_8, then $\mathrm{cd}(Q_8) = \mathrm{cd}(S_3)$ but $Q_8 \not\cong S_3 \times A$ for any abelian group A.

To give some evidence, Huppert verified the conjecture on a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type (cf. [11]). There has

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been much recent work in verifying the conjecture for simple groups of Lie type (cf. [11, 12, 25, 26, 28, 29, 30, 31] and [20, 27]). In particular, the conjecture has been verified for all simple groups of Lie type of Lie rank at most 2.

In this paper, we establish the Hupper's conjecture for the simple linear groups $\mathrm{PSL}_4(q)$. The case $\mathrm{PSL}_4(2)$, which is isomorphic to A_8, is already done by Huppert in [13]. It turns out that the proof in the case $3 \leq q \leq 11$ requires some ad hoc arguments and, as the paper is long enough, we decide to postpone these exceptional cases to another time. Hence we assume $q \geq 13$ from now on.

Theorem 1.1. *Let $q \geq 13$ be a prime power and let G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSL}_4(q))$. Then $G \cong \mathrm{PSL}_4(q) \times A$, where A is an abelian group.*

Our proof requires modifications in the five steps outlined by Huppert in [11] and relies heavily on the data given in Lubeck's webpage [18] on the complex character degrees of finite groups of Lie type. After a collection of some useful lemmas in section 2 and some analysis of character degrees of $\mathrm{PSL}_4(q)$ in section 3, we prove in section 4 that G is quasi-perfect, i.e. $G' = G''$. We then show in section 5 that if G'/M is a chief factor of G then $G'/M \cong \mathrm{PSL}_4(q)$. It is then shown in section 6 that if $C/M = C_{G/M}(G'/M)$ then $G/M \cong G'/M \times C/M$, which basically means that no outer automorphism of $\mathrm{PSL}_4(q)$ is involved in the structure of G. In section 7, we prove a technical result that every linear character of M is G'-invariant, which in particular implies that $M' = [M, G']$ and $|M : M'| \mid |\mathrm{Mult}(G'/M)|$ by Lemma 2.8. Using this, we deduce in section 8 that $M = 1$ and therefore $G' \cong \mathrm{PSL}_4(q)$ and $G \cong G' \times C_G(G')$, as desired. On the way to the proof of the main theorem, we also derive some interesting results on the behavior of the irreducible characters of $\mathrm{PSL}_4(q)$ under the action of its outer automorphism groups (see section 6).

Notation. Our notation is fairly standard (see, e.g. [5] and [14]). In particular, if G is a finite group then $\mathrm{Irr}(G)$ denotes the set of all irreducible characters of G and $\mathrm{cd}(G)$ denotes the set of degrees of irreducible characters in $\mathrm{Irr}(G)$. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. If $N \trianglelefteq G$ and $\lambda \in \mathrm{Irr}(N)$, then the induction of λ from N to G is denoted by λ^G and the set of irreducible constituents of λ^G is denoted by $\mathrm{Irr}(G|\lambda)$. Furthermore, if $\chi \in \mathrm{Irr}(G)$ then χ_N is the restriction of χ to N. The Schur multiplier of a group G is denoted by $\mathrm{Mult}(G)$. Finally, the greatest common divisor of two integers a and b is (a, b).

2. Preliminaries

In this section, we collect and also prove some lemmas which will be used throughout the paper.

Lemma 2.1. ([14, Corollary 11.29]). *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let ψ be a constituent of χ_N. Then $\chi(1)/\psi(1)$ divides $|G : N|$.*

Lemma 2.2 (Gallagher)**.** *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. If $\chi_N \in \mathrm{Irr}(N)$ then $\chi\tau \in \mathrm{Irr}(G)$ for every $\tau \in \mathrm{Irr}(G/N)$.*

Lemma 2.3 (Thompson)**.** *Suppose that p is a prime and $p \mid \chi(1)$ for every nonlinear $\chi \in \mathrm{Irr}(G)$. Then G has a normal p-complement.*

Lemma 2.4 ([14], Lemma 12.3 and Theorem 12.4)**.** *Let $N \lhd G$ be maximal such that G/N is solvable and nonabelian. Then one of the following holds.*

- (i) *G/N is a r-group for some prime r. If $\chi \in \mathrm{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$.*
- (ii) *G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Thus $|G : F| \in \mathrm{cd}(G)$, $|F : N| = r^a$ where a is the smallest integer such that $|G : F| \mid r^a - 1$. For every $\psi \in \mathrm{Irr}(F)$, either $|G : F|\psi(1) \in \mathrm{cd}(G)$ or $|F : N| \mid \psi(1)^2$. If no proper multiple of $|G : F|$ is in $\mathrm{cd}(G)$, then $\chi(1) \mid |G : F|$ for all $\chi \in \mathrm{Irr}(G)$ such that $r \nmid \chi(1)$.*

Lemma 2.5. *In the context of (ii) of Lemma 2.4, we have*

- (i) *If $\chi \in \mathrm{Irr}(G)$ such that $\mathrm{lcm}(\chi(1), |G : F|)$ does not divide any character degree of G, then $r^a \mid \chi(1)^2$.*
- (ii) *If $\chi \in \mathrm{Irr}(G)$ such that no proper multiple of $\chi(1)$ is a degree of G, then either $|G : F| \mid \chi(1)$ or $r^a \mid \chi(1)^2$*

Proof. (i) Suppose that ψ is a constituent of χ_F. Lemma 2.1 then implies that $\chi(1)/\psi(1) \mid |G : F|$. In other words, $\chi(1) \mid |G : F|\psi(1)$. Since $\mathrm{lcm}(\chi(1), |G : F|)$ does not divide any character degree of G, we see that $|G : F|\psi(1)$ is not a degree of G. This forces $r^a \mid \psi(1)^2$ by Lemma 2.4(ii). Since $\psi(1)$ divides $\chi(1)$, it follows that r^a divides $\chi(1)^2$.

(ii) Let $\psi \in \mathrm{Irr}(F)$ be an irreducible constituent of χ_F. By Lemma 2.4(ii), either $|G : F|\psi(1) \in \mathrm{cd}(G)$ or $r^a \mid \psi(1)^2$. If the latter case holds then we are done. So we assume that $|G : F|\psi(1) \in \mathrm{cd}(G)$. Write $\chi(1) = k\phi(1)$ for some integer k. It follows by Lemma 2.1 that $k \mid |G : F|$. Therefore, $|G : F|\chi(1)/k = |G : F|\phi(1) \in \mathrm{cd}(G)$. As no proper multiple of $\chi(1)$ is in $\mathrm{cd}(G)$, it follows that $|G : F| = k$ and hence $|G : F| \mid \chi(1)$. □

Lemma 2.6. ([2, Lemma 5]). *Let $N = S \times \cdots \times S$, a direct product of k copies of a nonabelian simple group S, be a minimal normal subgroup of K. If $\chi \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\chi(1)^k$ is a character degree of K.*

Proof. Note that N can be considered as a subgroup of $K/C_K(N)$ and $K/C_K(N)$ is embedded in $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr S_k$. Let λ be an extension of χ to $\mathrm{Aut}(S)$. Since $\lambda \times \cdots \times \lambda$ is invariant under $\mathrm{Aut}(S) \wr S_k$, it is extendable to $\mathrm{Aut}(S) \wr S_k$. Hence the character $\chi \times \cdots \times \chi \in \mathrm{Irr}(N)$ is extendable to $\mathrm{Aut}(S) \wr S_k$. In particular, it can be extended to $K/C_K(N)$. The lemma follows. □

Lemma 2.7. ([11, Lemma 3]). *Let $M \trianglelefteq G$, $\theta \in \mathrm{Irr}(M)$, and $I = I_G(\theta)$. If $\phi \in \mathrm{Irr}(I)$ lying above θ, then $\phi = \theta_0\tau$, where θ_0 is a character of an irreducible projective representation of I of degree $\theta(1)$ and τ is a character of an irreducible projective representation of I/M.*

Lemma 2.8. ([11, Lemma 6]). *Suppose $M \trianglelefteq G' = G''$ and $\lambda^g = \lambda$ for all $g \in G'$ and $\lambda \in \mathrm{Irr}(M)$ such that $\lambda(1) = 1$. Then $M' = [M, G']$ and $|M : M'|$ divides the order of the Schur multiplier of G'/M.*

The following result is an easy consequence of [14, Theorem 6.18].

Lemma 2.9. *Let $M \trianglelefteq L \trianglelefteq G$ be normal subgroups of a group G such that L/M is an abelian chief factor of G. Let $\theta \in \mathrm{Irr}(M)$ such that θ is L-invariant and let λ be an irreducible constituent of θ^L. Suppose that $\lambda(1) > \theta(1)$ and λ is G-invariant. Then $\lambda(1)/\theta(1) = \sqrt{|L : M|}$.*

Proof. As θ is L-invariant, by Clifford theory, we have that $\lambda_M = e\theta$ for some integer e. Since $\lambda(1) > \theta(1)$, we deduce that $e > 1$. By [14, Theorem 6.18] we obtain that $e^2 = |L : M|$ and so $\lambda(1) = e\theta(1) = \sqrt{|L : M|}\theta(1)$ as required. □

Lemma 2.10. *Let S be a nonabelian simple group. Let G be a perfect group so that $G/M \cong S$ and $|M| = |\mathrm{Mult}(S)|$ where M is cyclic. Then G is isomorphic the Schur cover of S.*

Proof. As M is normal in $G = G'$ and M is cyclic, $C_G(M)$ is normal in G and contains M. Since G/M is simple, we obtain that $C_G(M) = G$ or $C_G(M) = M$. If the former case holds, then $M \leq Z(G) \cap G'$ so that by definition G is the Schur cover of G/M. For the latter possibility, $G/C_G(M) = G/M$ embeds into $\mathrm{Aut}(M)$, which is solvable while G/M is simple, a contradiction. □

3. Character degrees of the simple linear groups $\mathrm{PSL}_4(q)$

Let Φ_k denote the kth cyclotomic polynomial evaluated at q. In particular,

$$\Phi_1 = q - 1, \Phi_2 = q + 1, \Phi_3 = q^2 + q + 1, \text{ and } \Phi_4 = q^2 + 1.$$

The data in [18] gives the character degrees of $\mathrm{SL}_4(q)$ and $\mathrm{PGL}_4(q)$. From there, we are able to extract the character degrees of $\mathrm{PSL}_4(q)$. These degrees are given in Table 1. The word "possible" in the second column means that the condition for the existence of corresponding degree is fairly complicated.

We establish some arithmetic properties of character degrees of $\mathrm{PSL}_4(q)$, which will be needed in sections 4 and 5. Recall that a power is nontrivial power if it has exponent greater than one. Also, a *primitive prime divisor* of Φ_k is a prime divisor of Φ_k that does not divide Φ_i for every $1 \leq i < k$. This prime exists by the classical result of Zsigmondy in [32].

TABLE 1. Character degrees of $\mathrm{PSL}_4(q)$ (see [18]).

Degrees	Conditions	Degrees	Conditions
1	any q	$q^2\Phi_1^2\Phi_3$	any q
$q\Phi_3$	any q	$\frac{1}{2}q^2\Phi_1^2\Phi_3$	possible
$\Phi_2\Phi_4$	$q=4$ or $q\geq 6$	$\Phi_1^2\Phi_3\Phi_4$	$q\geq 4$
$\Phi_1^2\Phi_3$	any q	$c\Phi_1^2\Phi_3\Phi_4$	$c=1/2,1/4$, possible
$\frac{1}{2}\Phi_1^2\Phi_3$	possible	$\Phi_1^2\Phi_2^2\Phi_4$	any q
$q^2\Phi_4$	any q	q^6	any q
$\Phi_3\Phi_4$	$q\geq 4$	$q\Phi_1\Phi_3\Phi_4$	any q
$\frac{1}{2}\Phi_3\Phi_4$	possible	$\Phi_1\Phi_2\Phi_3\Phi_4$	$q\geq 3$
$\Phi_1\Phi_3\Phi_4$	any q	$\frac{1}{2}\Phi_1\Phi_2\Phi_3\Phi_4$	possible
$q^3\Phi_3$	any q	$q^3\Phi_2\Phi_4$	$q=4$ or $q\geq 6$
$q\Phi_3\Phi_4$	$q\geq 3$	$q^2\Phi_3\Phi_4$	$q\geq 3$
$q\Phi_2^2\Phi_4$	$q=4$ or $q\geq 6$	$\frac{1}{2}q^2\Phi_3\Phi_4$	possible
$\Phi_2\Phi_3\Phi_4$	$q\geq 4$	$q\Phi_2\Phi_3\Phi_4$	$q\geq 4$
$\Phi_1^3\Phi_2\Phi_3$	any q	$\Phi_2^2\Phi_3\Phi_4$	$q\geq 7$
$c\Phi_1^3\Phi_2\Phi_3$	$c=1/2,1/4$, possible	$c\Phi_2^2\Phi_3\Phi_4$	$c=1/2,1/4$, possible

Lemma 3.1. *Let $\ell_i, i=1,2$ be primitive prime divisors of Φ_3 and Φ_4, respectively. If $\chi\in\mathrm{Irr}(\mathrm{PSL}_4(q))$ and $(\chi(1),\ell_1\ell_2)=1$, then $\chi(1)=q^6$.*

Proof. This is a straightforward check from Table 1. □

Lemma 3.2. *The degrees $\Phi_1^3\Phi_2\Phi_3$ and $\Phi_1^2\Phi_2^2\Phi_4$ are maximal with respect to divisibility among the degrees of $\mathrm{PSL}_4(q)$.*

Proof. The degree $\Phi_1^3\Phi_2\Phi_3$ is maximal with respect to divisibility among the degrees of $\mathrm{PSL}_4(q)$ since $(q,\Phi_1^3\Phi_2\Phi_3)=1$ and $(\Phi_4,\Phi_1^3\Phi_2\Phi_3)\mid 2$. Also, the degree $\Phi_1^2\Phi_2^2\Phi_4$ is maximal with respect to divisibility among degrees of $\mathrm{PSL}_4(q)$ since $(q,\Phi_1^2\Phi_2^2\Phi_4)=(\Phi_3,\Phi_1^2\Phi_2^2\Phi_4)=1$. □

Lemma 3.3. *The following assertions hold.*

(i) *If $\chi(1)$ and $\psi(1)$ are nontrivial degrees of $\mathrm{PSL}_4(q)$ such that $(\chi(1),\psi(1))=1$, then the set $\{\chi(1),\psi(1)\}$ contains at least one of the degrees: $q\Phi_3$, $\Phi_1^2\Phi_3$, $\frac{1}{2}\Phi_1^2\Phi_3$, $q^2\Phi_4$, $q^3\Phi_3$, $q^2\Phi_1^2\Phi_3$, $\frac{1}{2}q^2\Phi_1^2\Phi_3$, q^6.*

(ii) *The only pairs of consecutive integers that are degrees of $\mathrm{PSL}_4(q)$ are $(q\Phi_3, \Phi_2\Phi_4)$ (when $q=4$ or $q\geq 6$) and $(20,21)$ (when $q=2$).*

Proof. (i) Assume by contradiction that neither $\chi(1)$ nor $\psi(1)$ is one of the degrees $q\Phi_3$, $\Phi_1^2\Phi_3$, $\frac{1}{2}\Phi_1^2\Phi_3$, $q^2\Phi_4$, $q^3\Phi_3$, $q^2\Phi_1^2\Phi_3$, $\frac{1}{2}q^2\Phi_1^2\Phi_3$, q^6. Assume furthermore that both of them are not in $\{c\Phi_1^3\Phi_2\Phi_3 \mid c=1,1/2,1/4\}$. From the list of degrees of $\mathrm{PSL}_4(q)$ we see that both $\{\chi(1)$ and $\psi(1)\}$ are divisible by Φ_4, which violates the hypothesis.

So at least one of $\{\chi(1)$ and $\psi(1)\}$ is in $\{c\Phi_1^3\Phi_2\Phi_3 \mid c = 1, 1/2, 1/4\}$. A routine check gives the result.

(ii) This is obvious from Table 1. $\square$

Lemma 3.4. *The following assertions hold.*

(i) *If $\chi \in \mathrm{Irr}(\mathrm{PSL}_4(q))$ and $\Phi_3 \mid \chi(1)$ then $\chi(1)$ is not a nontrivial power.*

(ii) *The only possible degrees of $\mathrm{PSL}_4(q)$ which are nontrivial powers are $\Phi_2\Phi_4$, $q\Phi_2^2\Phi_4$, $\Phi_1^2\Phi_2^2\Phi_4$, q^6, $q^3\Phi_2\Phi_4$. In particular, $\mathrm{PSL}_4(q)$ has at most five nontrivial power degrees.*

Proof. (i) Remark that

$$(q, \Phi_3) = (\Phi_2, \Phi_3) = (\Phi_4, \Phi_3) = 1$$

and

$$(\Phi_1, \Phi_3) = 1 \text{ or } 3.$$

Therefore, if $\Phi_3 \mid \chi(1)$ and $\chi(1)$ is a nontrivial power then either Φ_3 or $\Phi_3/3$ is a nontrivial power. The former is impossible by [8] and the latter is impossible by [23].

(ii) This is a corollary of (i) and the fact that $q^2\Phi_4 = q^2(q^2 + 1)$ is not a nontrivial power by [8]. $\square$

Lemma 3.5. *If q is odd then*

$$\Phi_1\Phi_2\Phi_3\Phi_4/2 \in \mathrm{cd}(\mathrm{SL}_4(q)) \textit{ but } \Phi_1\Phi_2\Phi_3\Phi_4/2 \notin \mathrm{cd}(\mathrm{PSL}_4(q)).$$

Proof. Consider a semisimple element $s \in \mathrm{GL}_4(q)$ with eigenvalues

$$\alpha, -\alpha, \omega^{(q+1)/2}, -\omega^{(q+1)/2},$$

where $\alpha \in \mathbb{F}_q \backslash \{0\}$ and w is a generator of $\mathbb{F}_{q^2} \backslash \{0\}$. Then

$$C_{\mathrm{GL}_4(q)} \cong \mathrm{GL}_1(q) \times \mathrm{GL}_1(q) \times \mathrm{GL}_1(q^2).$$

Moreover, $\overline{s} = sZ(\mathrm{GL}_4(q))$ is a semisimple element of $\mathrm{PGL}_4(q)$. Note that if $\overline{t} \in C_{\mathrm{PGL}_4(q)}(\overline{s})$ then $ts = st$ or $ts = -st$. Therefore, $|C_{\mathrm{PGL}_4(q)}(\overline{s})| = 2|C_{\mathrm{GL}_4(q)}|/(q-1)$. The semisimple character $\chi_{\overline{s}} \in \mathrm{Irr}(\mathrm{SL}_4(q))$ associated to the conjugacy class of $\overline{s}$ has degree

$$\chi_{\overline{s}}(1) = \frac{|\mathrm{SL}_4(q)|_{p'}}{|C_{\mathrm{PGL}_4(q)}(\overline{s})|_{p'}} = \frac{\Phi_1\Phi_2\Phi_3\Phi_4}{2}.$$

Indeed, every irreducible character of $\mathrm{SL}_4(q)$ of degree $\Phi_1\Phi_2\Phi_3\Phi_4/2$ is constructed in this way. Remark that the determinant of s is $\alpha^2\omega^{q+1}$. As ω^{q+1} is a generator of $\mathbb{F}_q \backslash \{0\}$, $\alpha^2\omega^{q+1} \neq 1$. In particular, $\overline{s} \notin \mathrm{PGL}_4(q)'$ and hence $Z(\mathrm{SL}_4(q)) \notin \mathrm{Ker}(\chi_{\overline{s}})$. We conclude that $\chi_{\overline{s}} \notin \mathrm{Irr}(\mathrm{PSL}_4(q))$, which implies that $\Phi_1\Phi_2\Phi_3\Phi_4/2 \notin \mathrm{cd}(\mathrm{PSL}_4(q))$. $\square$

4. G is quasi-perfect: $G' = G''$

Recall that G is a finite group with the same character degree set as $H = \mathrm{PSL}_4(q)$. First, we show that $G' = G''$. Assume by contradiction that $G' \neq G''$ and let $N \lhd G$ be maximal such that G/N is solvable and nonabelian. By Lemma 2.4, G/N is an r-group for some prime r or G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N.

Case 1. G/N is an r-group for some prime r. Since G/N is nonabelian, there is $\theta \in \mathrm{Irr}(G/N)$ such that $\theta(1) = r^b > 1$. From the classification of prime power degree representations of quasi-simple groups in [19], we deduce that $\theta(1) = r^b$ must be equal to the degree of the Steinberg character of H of degree q^6 and thus $r^b = q^6$, which implies that $r = p$. By Lemma 2.3, G possesses a nontrivial irreducible character χ with $p \nmid \chi(1)$. Lemma 2.1 implies that $\chi_N \in \mathrm{Irr}(N)$. Using Gallagher's lemma, we deduce that $\chi(1)\theta(1) = q^6\chi(1)$ is a degree of G, which is impossible.

Case 2. G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N. Thus $|G : F| \in \mathrm{cd}(G)$, $|F : N| = r^a$ where a is the smallest integer such that $|G : F| \mid r^a - 1$. Let χ be a character of G of degree q^6. As no proper multiple of q^6 is in $\mathrm{cd}(G)$, Lemma 2.5 implies that either $|G : F| \mid q^6$ or $r = p$. We consider two following subcases.

(a) $|G : F| \mid q^6$. Then $|G : F| = q^6$ by Table 1. This means no multiple of $|G : F|$ is in $\mathrm{cd}(G)$. Therefore, by Lemma 2.4, for every $\psi \in \mathrm{Irr}(G)$ either $\psi(1) \mid q^6$ or $r \mid \psi(1)$. Taking ψ to be characters of degrees $q\Phi_3$ and $q^2\Phi_4$, we obtain that r divides both Φ_3 and Φ_4. This leads to a contradiction since $(\Phi_3, \Phi_4) = 1$.

(b) $r = p$. By Lemma 3.2 and the fact that $p \nmid \Phi_1^3\Phi_2\Phi_3$ as well as $\Phi_1^2\Phi_2^2\Phi_4$, we have $|G : F|$ divides both $\Phi_1^3\Phi_2\Phi_3$ and $\Phi_1^2\Phi_2^2\Phi_4$. It follows that $|G : F|$ is prime to $\ell_1\ell_2$ so that by Lemma 3.1, we have $|G : F| = q^6$ as $|G : F| > 1$, which is impossible as $|G : F|$ is prime to p.

5. Eliminating the finite simple groups other than $\mathrm{PSL}_4(q)$

We have seen from section 4 that $G' = G''$. Moreover, it is easy to see that G' is nontrivial. Therefore, if G'/M is a chief factor of G, then $G'/M \cong S \times \cdots \times S$, a direct product of k copies of a nonabelian simple group S. As G'/M is a minimal normal subgroup of G/M and $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSL}_4(q))$, it follows by Proposition 5.1 below that $k = 1$ and $S \cong \mathrm{PSL}_4(q)$. Equivalently, $G'/M \cong \mathrm{PSL}_4(q)$, as we wanted.

Proposition 5.1. *Let K be a group. Suppose that $\mathrm{cd}(K) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$ and $N = S \times \cdots \times S$, a direct product of k copies of a nonabelian simple group S, is a minimal normal subgroup of K. Then $k = 1$ and $S \cong \mathrm{PSL}_4(q)$.*

The rest of this section is devoted to the proof of this proposition. Indeed, the proof is the combination of Lemmas 5.3, 5.4, 5.5, and 5.6. First, we need the following.

Lemma 5.2. *If S is a sporadic group or the Tits group, then there are at least six nontrivial irreducible characters of S of distinct degrees which are extendable to* $\mathrm{Aut}(S)$.

Proof. This can be checked by using Atlas [5]. □

Lemma 5.3. *With the hypothesis of Proposition 5.1, S is not an alternating group of degree larger than* 6.

Proof. Assume that $S = A_n$ with $n \geq 7$. It is well-known that the irreducible characters of S_n are in one-to-one correspondence with partitions of size n. In particular, S_n has three degrees $\chi_1(1) = n(n-3)/2$, $\chi_2(1) = (n-1)(n-2)/2$ and $\chi_3(1) = n-1$ corresponding to partitions $(2^2, 1^{n-4})$, $(3, 1^{n-3})$ and $(n-1, 1)$, respectively. As $n \geq 7$, these partitions are not self-conjugate and hence $\chi_i, i = 1, 2, 3$ are still irreducible when restricting to A_n. Using Lemma 2.6, we deduce that $\chi_i(1)^k \in \mathrm{cd}(K)$ and hence $\chi_i(1)^k \in \mathrm{cd}(\mathrm{PSL}_4(q))$ for $i = 1, 2, 3$. Since $(\chi_1(1)^k, \chi_2(1))^k = 1$, Lemma 3.3(i) implies that the set $\{\chi_1(1)^k, \chi_2(1)^k\}$ contains at least one of the degrees

$$q\Phi_3, \Phi_1^2\Phi_3, \frac{1}{2}\Phi_1^2\Phi_3, q^2\Phi_4, q^3\Phi_3, q^2\Phi_1^2\Phi_3, \frac{1}{2}q^2\Phi_1^2\Phi_3, q^6.$$

If $k \geq 2$, since q^6 is the only nontrivial power in this list by Lemma 3.4(i), we would have $q^6 \in \{\chi_1(1)^k, \chi_2(1)^k\} = \{(n(n-3)/2)^k, ((n-1)(n-2)/2)^k\}$. This is impossible when $n \geq 7$ and $n \neq 9$. Now assume that $n = 9$. In this case, we have that $\chi_3(1)^k = 8^k$ is a nontrivial prime power degree of $\mathrm{PSL}_4(q)$. However this is impossible by applying [19].

We have shown that the only choice for k is 1. That means the consecutive integers $n(n-3)/2$ and $(n-1)(n-2)/2$ are both degrees of $\mathrm{PSL}_4(q)$. Lemma 3.3(ii) now yields

$$n(n-3)/2 = q\Phi_3 \text{ and } q = 4 \text{ or } q \geq 6$$

or

$$n(n-3)/2 = 20 \text{ and } q = 2.$$

The latter case can be eliminated easily. So we assume that $n(n-3)/2 = q\Phi_3$ and $q \geq 4$. Since $\chi_3(1) = n-1 \in \mathrm{cd}(\mathrm{PSL}_4(q))$ and $n-1 < n(n-3)/2$ and $n(n-3/2) = q\Phi_3$ is the smallest nontrivial degree of $\mathrm{PSL}_4(q)$, we obtain a contradiction. □

Lemma 5.4. *With the hypothesis of Proposition 5.1, S is not a sporadic simple group nor the Tits group.*

Proof. Assume that S is a simple sporadic group or Tits group. By Lemma 5.2, S has at least six nontrivial irreducible characters of distinct degrees which are extendable to $\mathrm{Aut}(S)$. The kth powers of these degrees will be degrees of K by Lemma 2.6. As $\mathrm{cd}(K) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$, it follows that $\mathrm{cd}(\mathrm{PSL}_4(q))$ contains at least six nontrivial powers if $k \geq 2$, which is impossible by Lemma 3.4(ii).

We have shown that if S is a simple sporadic group or Tits group then $k = 1$.

If S is one of Ly, Th, Fi'_{24}, B, or M, then by Atlas again, we see that S has at least 33 nontrivial irreducible characters of distinct degrees which extend to $\mathrm{Aut}(S)$. This is a contradiction since $\mathrm{PSL}_4(q)$ has at most 32 nontrivial degrees.

Assume next that S is $O'N$. By [5], $O'N$ has an irreducible character degree 10,944 which is extendable to $\mathrm{Aut}(S)$. Therefore, as $q\Phi_3$ is the smallest nontrivial degree of $\mathrm{cd}(\mathrm{PSL}_4(q))$, we have $q\Phi_3 \leq 10{,}944$. It follows that $13 \leq q \leq 19$ and these cases can be ruled out directly by using the fact that $\{19, 31\} \subseteq \pi(S) \subseteq \pi(\mathrm{PSL}_4(q))$.

For the remaining cases, by inspecting Atlas [5], we see that S has a nontrivial irreducible character of degree smaller than 2379 which is extendable to $\mathrm{Aut}(S)$. This degree is a degree of $\mathrm{PSL}_4(q)$ by Lemma 2.6, which is a contradiction as the smallest nontrivial degree of $\mathrm{PSL}_4(q)$ is $q\Phi_3$, which is at least $13(13^2 + 13 + 1) = 2379$. $\square$

Lemma 5.5. *With the hypothesis of Proposition 5.1, S is not a simple Lie type group of classical type except* $\mathrm{PSL}_4(q)$. *Furthermore, if $S \cong \mathrm{PSL}_4(q)$ then $k = 1$.*

Proof. Suppose that S is a simple group of Lie type $G(q_1)$ where $q_1 = r^b$, a prime power. Let St denote the Steinberg character of S. It is well-known (see [9] for instance) that St is extendable to $\mathrm{Aut}(S)$ and $\mathrm{St}(1) = |S|_r$, the r-part of $|S|$. By Lemma 2.6, we get $|S|_r^k \in \mathrm{cd}(\mathrm{PSL}_4(q))$. From the classification of irreducible representations of quasi-simple groups of prime power degrees in [19, Theorem 1.1], we have that q^6 is the only degree of $\mathrm{PSL}_4(q)$ of prime power and therefore $|S|_r^k = q^6$ and $r = p$.

Consider a nontrivial character $\tau \in \mathrm{Irr}(S)$ different from St. Then $\tau \times \mathrm{St} \times \cdots \times \mathrm{St} \in \mathrm{Irr}(N)$ and hence $\tau(1)\mathrm{St}(1)^{k-1} = \tau(1)|S|_r^{k-1}$ divides some degree of K. Inspecting the list of degrees of $\mathrm{PSL}_4(q)$, we then deduce that $|S|_r^{k-1} \leq q^3$. This together with the fact $|S|_r^k = q^6$ imply that $k \leq 2$.

We wish to show that $k = 1$. By way of contradiction, assume that $k = 2$. Hence $|S|_p^2 = q^6$. Let $C = C_K(N)$. Then K/C embeds into $\mathrm{Aut}(N) \cong \mathrm{Aut}(S) \wr \mathbb{Z}_2$. Let $B = \mathrm{Aut}(S)^2 \cap K$. Then $K/B \cong \mathbb{Z}_2$. Let $\varphi = 1 \times St \in \mathrm{Irr}(N)$. We have that φ extends to B and so B is the inertia group of φ in K. It follows that $\varphi^K(1) = 2\varphi(1) = 2|S|_p$ is a degree of K and then $2|S|_p \in \mathrm{cd}(H)$. As this degree is prime to $\ell_1\ell_2$, Lemma 3.1 implies that $2|S|_p = |S|_p^2$. It follows that $|S|_p = 2$, which is impossible.

We consider the following cases. In all cases except $\mathrm{PSL}_4(q)$, we will reach a contradiction by finding a character $\chi \in \mathrm{Irr}(S)$ extendable to $\mathrm{Aut}(S)$ so that $\chi(1) \notin \mathrm{cd}(\mathrm{PSL}_4(q))$ or a character $\psi \in \mathrm{Irr}(S)$ so that $\psi(1)$ divides no degree of $\mathrm{PSL}_4(q)$.

(i) $S \cong \mathrm{PSL}_n^{\pm}(q_1)$ where $q_1 = p^b$. Then we get

$$q^6 = |S|_p = p^{bn(n-1)/2}. \tag{5.1}$$

From Table 2, S has a unipotent character χ different from the Steinberg character with $\chi(1)_p = p^{b(n-1)(n-2)/2}$. The fact that $\chi(1) \in \mathrm{cd}(\mathrm{PSL}_4(q))$ and Table 1 then yield

$$p^{b(n-1)(n-2)/2} \mid q^3 \tag{5.2}$$

TABLE 2. Some unipotent characters of simple groups of Lie type [3].

$S = G(q)$	p-part of degrees
$\mathrm{PSL}_n^{\pm}(q)$	$q^{(n-1)(n-2)/2}$
$\mathrm{PSp}_{2n}(q)$ or $\mathrm{P}\Omega_{2n+1}(q)$, $p = 2$	$q^{(n-1)^2}/2$
$\mathrm{PSp}_{2n}(q)$ or $\mathrm{P}\Omega_{2n+1}(q)$, $p > 2$	$q^{(n-1)^2}$
$\mathrm{P}\Omega_{2n}^{+}(q)$	q^{n^2-3n+3}
$\mathrm{P}\Omega_{2n}^{-}(q)$	q^{n^2-3n+2}
${}^3D_4(q)$	q^7
${}^2F_4(q)$	$q^6\sqrt{q/2}$
${}^2E_6(q)$	q^{25}
$F_4(q)$, $p = 2$	$q^{13}/2$
$F_4(q)$, $p > 2$	q^{13}
$E_6(q)$	q^{25}
$E_7(q)$	q^{46}
$E_8(q)$	q^{91}

Now (5.1) and (5.2) imply $2(n-2) \leq n$ and hence $n \leq 4$.

If $n = 2$ then $q_1 = q^6$. The fact that $\mathrm{PSL}_2(q_1)$ has an irreducible of degree $q_1 + 1$ (see [18]) then leads to a contradiction since both q^6+1 divides no degree of $PSL_4(q)$. If $n = 3$ then $q_1^3 = q^6$. We see that $q_1 = q^2$. This case does not happen since $\mathrm{PSL}_3^{\pm}(q^2)$ has an irreducible character of degree $(q^2 \mp 1)^2(q^2 \pm 1)$ which does not divide any degree of $\mathrm{PSL}_4(q)$. Finally, if $n = 4$ then $q_1^6 = q^6$ and hence $q_1 = q$. The case $S \cong \mathrm{PSU}_4(\mathrm{q})$ does not happen since $\mathrm{PSU}_4(q)$ has a unipotent character of degree $q^3\Phi_6 \notin \mathrm{cd}(\mathrm{PSL}_4(q))$ (see [4, Table 20]). We are left with the case $S \cong \mathrm{PSL}_4(q)$, as wanted.

(ii) $S \cong \mathrm{P}\Omega_{2n+1}(q_1)$ or $\mathrm{PSp}_{2n}(q_1)$ where $q_1 = p^b$ and $n \geq 2$. Then we get

$$q^6 = |S|_p = p^{bn^2}. \tag{5.3}$$

From Table 2, S has a unipotent character χ different from the Steinberg character with $\chi(1)_p = p^{b(n-1)^2-1}$ when $p = 2$ or $\chi(1)_p = p^{b(n-1)^2}$ when $p > 2$. The fact that $\chi(1) \in \mathrm{cd}(\mathrm{PSL}_4(q))$ and Table 1 then yield

$$p^{(b(n-1)^2-1)} \mid q^3 \tag{5.4}$$

Now (5.3) and (5.4) imply $2(b(n-1)^2 - 1) \leq bn^2$ and hence $n \leq 4$.

If $n = 2$ then $q_1^4 = q^6$ and hence $q_1^2 = q^3$. Therefore $q_1 = q^{3/2}$. This case cannot happen since $\mathrm{PSp}_4(q_1)$ has a unipotent character of degree $q_1(q_1-1)^2/2$ (see [4, Table 23]) and $q_1^2(q_1-1)^2/2 = q^3(q^{3/2}-1)^2/2 \notin \mathrm{cd}(\mathrm{PSL}_4(q))$. If $n = 3$ then $q_1^9 = q^6$. We see that $q_1 = q^{2/3}$. Taking a unipotent character of S of degree $q_1(q_1-1)^2(q_1^2+q_1+1)$ (see [4, Table 24]) and arguing as in the case $n = 2$, we get a contradiction. Finally,

if $n = 4$ then $q_1^{16} = q^6$. We see that $q_1 = q^{3/8}$. Taking a unipotent character of S of degree $q_1(q_1^2 - q_1 + 1)(q_1^4 + 1)$ (see [4, Table 25]) and arguing as in the case $n = 2$, we get a contradiction.

(iii) $S \cong P\Omega_{2n}^{\pm}(q_1)$ where $q_1 = p^b$ and $n \geq 4$. Then we get

$$q^6 = |S|_p = p^{bn(n-1)}. \tag{5.5}$$

From Table 2, S has a unipotent character χ different from the Steinberg character with $\chi(1)_p = p^{b(n^2-3n+3)}$ when $S \cong \mathrm{P}\Omega_{2n}^{+}(q_1)$ and $\chi(1)_p = p^{b(n^2-3n+2)}$ when $S \cong \mathrm{P}\Omega_{2n}^{-}(q_1)$. The fact that $\chi(1) \in \mathrm{cd}(\mathrm{PSL}_4(q))$ and Table 1 then yield

$$p^{b(n^2-3n+2)} \mid q^3 \tag{5.6}$$

Now (5.5) and (5.6) imply $2(n^2 - 3n + 2) \leq n(n-1)$ and hence $n = 4$. Furthermore, $S \cong \mathrm{P}\Omega_8^-(q_1)$.

Since $n = 4$, we have $q_1^{12} = q^6$. Therefore, $q_1 = q^{1/2}$. Taking a unipotent character of S of degree $q_1(q_1^4 + 1)$ (see [4, Table 29]) and arguing as in the previous case, we get a contradiction. □

Lemma 5.6. *With the hypothesis of Proposition 5.1, S is not a simple Lie type group of exceptional type.*

Proof. It follows from the proof of the previous lemma that $k = 1$. First we eliminate the case $S \cong {}^2B_2(q_1) = {}^2B_2(2^{2m+1})$ where $m \geq 1$. Assume that $2^{2(2m+1)} = q^6$ and hence $2^{(2m+1)} = q^3$. Also, S has a unipotent character of degree $\sqrt{q_1/2}(q_1 - 1) = 2^m(2^{2m+1} - 1)$ and therefore $2^m(2^{2m+1} - 1) \in \mathrm{cd}(\mathrm{PSL}_4(q))$. It follows that $2^m \mid q^3$. Together with the equality $2^{2m+1} = q^3$, we obtain $m = 1$. This case can be ruled out easily by using Atlas. From now on we assume that S is one of the simple groups of exceptional Lie type different from ${}^2B_2$.

(i) $S \cong G_2(q_1)$ where $q_1 = p^b$. Then we have $q_1^6 = |S|_p = q^6$ and hence $q_1 = q$. That means $S \cong G_2(q)$ and S has a unipotent character of degree $q\Phi_3\Phi_6/3$ (see [4, Table 33]). However, this is not a degree of $\mathrm{PSL}_4(q)$.

(ii) $S \cong {}^2G_2(q_1) = {}^2G_2(3^{2m+1})$ where $m \geq 1$. Then we have $q_1^3 = |S|_p = q^6$ and hence $3^{2m+1} = q_1 = q^2$, which is impossible.

(iii) For the remaining groups, S has a unipotent character (see Table 2) different from the Steinberg character whose degree has p-part larger than $\sqrt{|S|_p} = \sqrt{q^6} = q^3$. By Lemma 2.6, this degree is also a degree of K and therefore a degree of $\mathrm{PSL}_4(q)$. However, all degrees different from q^6 of $\mathrm{PSL}_4(q)$ has p-part at most q^3 by Table 1. This leads to a contradiction. □

6. Outer automorphisms of $\mathrm{PSL}_4(q)$

In section 5, we have shown that $G'/M \cong \mathrm{PSL}_4(q)$. In this section, we will prove that

$$G/M \cong G'/M \times C/M,$$

where $C/M = C_{G/M}(G'/M)$. Remark that $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$. For simple notation, we can assume that $M = 1$ but only assume $\mathrm{cd}(G) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$. In other words, we have to prove the following:

Proposition 6.1. *Let G be a finite group such that $G' \cong \mathrm{PSL}_4(q)$ and $\mathrm{cd}(G) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$. Then $G \cong G' \times C_G(G')$.*

Proof. Assume the contrary that $G' \times C_G(G')$ is a proper subgroup of G. Then G induces on G' some outer automorphism α. Let $q = p^f$. It is well known (cf. [10, Theorem 2.5.12]) that

$$\mathrm{Out}(G') = \langle d \rangle : (\langle \sigma \rangle \times \langle \tau \rangle),$$

where d is a diagonal automorphism of degree $(q-1,4)$, σ is the automorphism of G' of order f induced by the field automorphism $x \mapsto x^p$, and τ is the inverse-transpose. Moreover, τ inverts the cyclic group $\langle d \rangle$.

(i) First we consider the case where $G/C_G(G')$ possesses only inner and diagonal automorphisms. This means q must be odd.

Case $\mathbf{q \equiv 3}$ mod $\mathbf{4}$: We then have

$$G/C_G(G') \cong \mathrm{PGL}_4(q).$$

From the list of degrees of $\mathrm{PGL}_4(q)$ and $\mathrm{SL}_4(q)$ in [18], we see that $\mathrm{PGL}_4(q)$ has $q-2$ irreducible characters of degree $\Phi_3\Phi_4$ while $\mathrm{SL}_4(q)$ has $(q-3)/2$ characters of degree $\Phi_3\Phi_4$ and 2 characters of degree $\Phi_3\Phi_4/2$. It follows that $\mathrm{PSL}_4(q)$ has exactly two irreducible characters of degree $\Phi_3\Phi_4/2$ and they are fused under $\mathrm{PGL}_4(q)$.

We will get to a contradiction by showing that $\Phi_3\Phi_4/2$ is not a degree of G. Assume by contrary that χ is an irreducible character of G of such degree. As $G/C_G(G') \cong \mathrm{PGL}_4(q)$, which has no irreducible character of degree $\Phi_3\Phi_4/2$, $\chi \notin \mathrm{Irr}(G/C_G(G'))$. Assume that $C_G(G')$ is not abelian and let λ be a nonlinear irreducible character of $C_G(G')$. Then $\mathrm{St}_{G'} \times \lambda \in \mathrm{Irr}(G' \times C_G(G'))$ where $\mathrm{St}_{G'}$ is the Steinberg character of $G' = \mathrm{PSL}_4(q)$. This is a contradiction since $\mathrm{St}_{G'}(1)\lambda(1) = q^6\lambda(1)$ divides none of the degree of G.

We have shown that $C_G(G')$ is abelian. Let $\psi \times \theta$ an irreducible character $G' \times C_G(G')$ lying under χ. As $C_G(G')$ is abelian, we know $\theta(1) = 1$. Assume that θ is not G-invariant. As G' centralizes $C_G(G')$ and $|G : G' \times C_G(G')| = 2$, it follows that $I_G(\theta) = G' \times C_G(G')$. Let $\theta_0 = 1_{G'} \times \theta \in \mathrm{Irr}(I_G(\theta))$. Then θ_0 is an extension of θ to $I_G(\theta)$, and therefore $\theta_0^G \in \mathrm{Irr}(G)$ by Clifford theory. So

$$\theta_0^G(1) = |G : (G' \times C_G(G'))|\theta_0(1) = 2$$

is a degree of G, which is impossible. Thus θ is G-invariant.

If ψ is G-invariant then ψ, considered as a character of $G'C_G(G')/C_G(G')$, is also $G/C_G(G')$-invariant. As $|G : G'C_G(G')| = 2$, we deduce from [14, Corollary 6.20] that ψ extends to $G/C_G(G')$. In particular,

$$\psi(1) \in \mathrm{cd}(G/C_G(G')) = \mathrm{cd}(\mathrm{PGL}_4(q)).$$

On the other hand, we also have that $\psi \times \theta$ is G-invariant and hence it extends to G. So

$$\chi(1) = (\psi \times \theta)(1) = \psi(1)$$

and therefore $\Phi_3\Phi_4/2$ is a degree of $\mathrm{PGL}_4(q)$, a contradiction. Thus ψ is not G-invariant and hence $\psi \times \theta$ is not G-invariant and $I_G(\psi \times \theta) = G' \times C_G(G')$. Using Clifford theory again, we obtain $(\psi \times \theta)^G \in \mathrm{Irr}(G)$ and hence

$$\chi(1) = (\psi \times \theta)^G(1) = 2\psi(1).$$

Recall that $\chi(1) = \Phi_3\Phi_4/2$. We then have $\psi(1) = \Phi_3\Phi_4/4$ is a degree of $G' = \mathrm{PSL}_4(q)$, which is impossible.

Case $\mathbf{q \equiv 1}$ mod $\mathbf{4}$: In this case, d is a diagonal automorphism of G' of degree 4. We then have

$$G/C_G(G') \cong G' : \langle d^2 \rangle \text{ or } \mathrm{PGL}_4(q).$$

From the list of degrees of $\mathrm{PGL}_4(q)$ and $\mathrm{SL}_4(q)$ in [18], we see that G' has exactly four irreducible characters of degree $\Phi_1^2\Phi_3\Phi_4/4$, which are fused under $G' : \langle d^2 \rangle$ to form two characters of degree $\Phi_1^2\Phi_3\Phi_4/2$ and under $\mathrm{PGL}_4(q)$ to form one character of degree $\Phi_1^2\Phi_3\Phi_4$.

We will get to a contradiction by showing that $\Phi_1^2\Phi_3\Phi_4/4$ is not a degree of G. Assume by contrary that χ is an irreducible character of G of such degree. As $G/C_G(G') \cong G' : \langle d^2 \rangle$ or $\mathrm{PGL}_4(q)$, which has no irreducible character of degree $\Phi_1^2\Phi_3\Phi_4/4$ as $q \neq 5$ (when $q = 5$, there is a coincidence $\Phi_1^2\Phi_3\Phi_4/4 = \Phi_1\Phi_3\Phi_4 \in \mathrm{cd}(\mathrm{PGL}_4(q))$), $\chi \notin \mathrm{Irr}(G/C_G(G'))$. Assume that $C_G(G')$ is not abelian and let λ be a nonlinear irreducible character of $C_G(G')$. Then

$$\mathrm{St}_{G'} \times \lambda \in \mathrm{Irr}(G' \times C_G(G')),$$

where $\mathrm{St}_{G'}$ is the Steinberg character of $G' = \mathrm{PSL}_4(q)$. This is a contradiction since $\mathrm{St}_{G'}(1)\lambda(1) = q^6\lambda(1)$ divides none of the degree of G.

We have shown that $C_G(G')$ is abelian. Let $\psi \times \theta$ be an irreducible character $G' \times C_G(G')$ lying under χ. As $C_G(G')$ is abelian, we know $\theta(1) = 1$. Assume that θ is not G-invariant. As G' centralizes $C_G(G')$ and $|G : G' \times C_G(G')| = 4$ or 2, it follows that

$$|I_G(\theta) : (G' \times C_G(G'))| = 1 \text{ or } 2 \text{ if } |G : G' \times C_G(G')| = 4$$

and

$$|I_G(\theta) : (G' \times C_G(G'))| = 1 \text{ if } |G : G' \times C_G(G')| = 2.$$

Let θ_0 be an extension of $1_{G'} \times \theta \in \mathrm{Irr}(G' \times C_G(G'))$ to $I_G(\theta)$. Then $\theta_0^G \in \mathrm{Irr}(G)$ by Clifford theory and hence

$$\theta_0^G(1) = |G : I_G(\theta)|\theta_0(1) = 2 \text{ or } 4,$$

which is a degree of G, a contradiction. Thus θ is G-invariant.

If ψ is G-invariant then ψ, considered as a character of $G'C_G(G')/C_G(G')$, is also $G/C_G(G')$-invariant. As $|G : G'C_G(G')|$ is cyclic, we deduce from [14, Corollary 11.22] that ψ extends to $G/C_G(G')$. In particular, $\psi(1) \in \mathrm{cd}(G/C_G(G'))$. On the other hand, we also have that $\psi \times \theta$ is G-invariant and hence it extends to G. So

$$\chi(1) = (\psi \times \theta)(1) = \psi(1).$$

This means $\Phi_1^2\Phi_3\Phi_4/4$ is a degree of $G/C_G(G')$, which is $G' : \langle d^2\rangle$ or $\mathrm{PGL}_4(q)$, a contradiction. So ψ is not G-invariant and hence $\psi \times \theta$ is not G-invariant and therefore

$$|I_G(\psi \times \theta) : G' \times C_G(G')| = 1 \text{ or } 2.$$

Let φ be an extension of $\psi \times \theta$ to $I_G(\psi \times \theta)$. Using Clifford theory, we obtain $(\varphi)^G \in \mathrm{Irr}(G)$ and hence

$$(\varphi)^G(1) = |G : I_G(G' \times C_G(G'))|\psi(1),$$

which is $2\psi(1)$ or $4\psi(1)$. Since χ lies over $\psi \times \theta$, it follows that $\chi(1) = 2\psi(1)$ or $4\psi(1)$. Recall that $\chi(1) = \Phi_1^2\Phi_3\Phi_4/4$. So $\psi(1) = \Phi_1^2\Phi_3\Phi_4/8$ or $\Phi_1^2\Phi_3\Phi_4/16$. This contradicts the fact that $\psi(1)$ is a degree of $G' = \mathrm{PSL}_4(q)$.

(ii) Next we consider the case $\alpha = d^a\sigma^b\tau^c$ where $0 \le a \le (q-1,4)$, $0 < b < f$, and $0 \le c \le 1$. By a classical result of Zsigmondy (cf. [32]), we can choose $\omega \in \mathbb{F}$ of order which is a primitive prime divisor of $p^{4f} - 1 = q^4 - 1$, that is a prime divisor of $p^{4f} - 1$ that does not divide $\prod_{i=1}^{4f-1}(p^i - 1)$. We then choose a semisimple element $s \in \mathrm{SL}_4(q)$ with eigenvalues $\omega, \omega^q, \omega^{q^2}$, and ω^{q^3}. The image of s under the canonical projection $\mathrm{GL}_4(q) \to \mathrm{PGL}_4(q)$ is a semisimple element of $\mathrm{PGL}_4(q)$. Abusing the notation, we denote it by s. Since s, s^{-1}, and $\tau(s)$ are all conjugate in $\mathrm{PGL}_4(q)$, the semisimple character $\chi_s \in \mathrm{Irr}(\mathrm{SL}_4(q))$ of degree $\Phi_1^3\Phi_2\Phi_3$ is real by [7, Lemma 2.5] and hence

$$(\chi_s)^\tau = \chi_{\tau(s)} = \chi_{s^{-1}} = \overline{\chi}_s = \chi_s$$

by [24, Corollary 2.5]. In other words, χ_s is invariant under τ.

Since $\mathrm{PGL}_4(q)$ has no degree which is a proper multiple of $\Phi_1^3\Phi_2\Phi_3$, we obtain that χ_s is also d-invariant. By checking the multiplicities of character degrees of $\mathrm{SL}_4(q)$ and $\mathrm{PGL}_4(q)$, we see that there exists an s as above so that $\chi_s \in \mathrm{Irr}(\mathrm{PSL}_4(q))$. Using [7, Lemma 2.5] again, we have furthermore that χ_s is not σ^b-invariant since $|s| = |\omega|$ does not divide $|\mathrm{PGL}_4(p^b)|$. We have shown that χ_s is not α-invariant. Therefore G has a degree which is a proper multiple of $\chi_s(1) = \Phi_1^3\Phi_2\Phi_3$, a contradiction.

(iii) Finally we consider the case $\alpha = d^a\tau$ where $0 \le a \le (q-1,4)$. Now we choose $\omega \in \mathbb{F}$ of order which is a primitive prime divisor of $p^{3f} = q^3 - 1$, that is

a prime divisor of $p^{3f}-1$ that does not divide $\prod_{i=1}^{3f-1}(p^i-1)$. We then choose a semisimple element $s \in \mathrm{SL}_4(q)$ with eigenvalues $1, \omega, \omega^q$, and ω^{q^2}. The image of s under the canonical projection $\mathrm{GL}_4(q) \to \mathrm{PGL}_4(q)$ is a semisimple element of $\mathrm{PGL}_4(q)$. Abusing the notation, we denote it by s. It is routine to check that ω^{-1} is not an eigenvalue of s. In particular, s is not conjugate to s^{-1} in $\mathrm{PGL}_4(q)$ and so s is not real in $\mathrm{PGL}_4(q)$. It follows by [7, Lemma 2.5] that the semisimple character $\chi_s \in \mathrm{Irr}(\mathrm{SL}_4(q))$ of degree $\Phi_1^2\Phi_2^2\Phi_4$ is not real. Moreover, χ_s is trivial at $Z(\mathrm{SL}_4(q))$ and therefore $\chi_s \in \mathrm{Irr}(\mathrm{PSL}_4(q))$. Now using [24, Corollary 2.5] and the fact that $\tau(s)$ is conjugate to s^{-1} in $\mathrm{PGL}_4(q)$, we get

$$(\chi_s)^\tau = \chi_{\tau(s)} = \chi_{s^{-1}} = \overline{\chi}_s.$$

Therefore χ_s is moved by τ since it is not real. This is a contradiction since G does not have any degree which is a proper multiple of $\chi_s(1) = \Phi_1^2\Phi_2^2\Phi_4$. □

7. Linear characters of M are G'-invariant

We assume in this section that $q \geq 13$. In this step, we prove that if $\theta \in \mathrm{Irr}(M)$ and $\theta(1) = 1$ then $I := I_{G'}(\theta) = G'$. Assume by contradiction that $I < G'$, then $I \leq U < G'$ for some maximal subgroup U of G'. Suppose that

$$\theta^I = \sum_i \phi_i, \text{ where } \phi_i \in \mathrm{Irr}(I).$$

We then have $\phi_i^{G'} \in \mathrm{Irr}(G')$ and hence $\phi_i(1)|G' : I| \in \mathrm{cd}(G')$. It follows that

$$\phi_i(1)|G' : U||U : I| \text{ divides some degree of } G.$$

For simpler notation, let $t = |U : I|$, we have

$$t\phi_i(1)|G' : U| \text{ divides some degree of } G.$$

In particular, the index of U/M in G'/M divides some degree of G.

The maximal subgroups of $\mathrm{PSL}_4(q)$ as well as other groups in this paper are determined by Kleidman in [16, 17] and given in the following tables. In these tables, $d = (q-1, 4)$ and the symbol ^ means we are giving the structure of the preimage in special linear or symplectic groups. Let τ be the natural projection from $\mathrm{SL}_4(q)$ to $\mathrm{PSL}_4(q)$. The following lemma will eliminate most of possibilities of U/M.

Lemma 7.1. *With the above notation, one of the following happens.*

(i) $U/M \cong P_a \cong \tau(\hat{}[q^3] : \mathrm{GL}_3(q))$, $|G' : U| = \Phi_2\Phi_4$ *and for every* i, $t\phi_i(1)$ *divides some member of the set* $\mathcal{A}_1$, *where*

$$\mathcal{A}_1 = \{q\Phi_2, \Phi_1^2\Phi_2, \Phi_1\Phi_3, q\Phi_3, q^3, \Phi_2\Phi_3\}.$$

(ii) $U/M \cong \tau(\hat{}\mathrm{Sp}_4(q) \cdot (q-1, 2))$, $|G' : U| = q^2\Phi_1\Phi_3/(2, q-1)$, *and for every* i, $t\phi_i(1)$ *divides* $(2, q-1)\Phi_1$.

TABLE 3. Maximal subgroups of $\mathrm{PSL}_4(q)$ (see [16]).

Subgroup	Condition	Index
$\hat{}[q^3]:\mathrm{GL}_3(q)$		$\Phi_2\Phi_4$
$\hat{}[q^4]:(\mathrm{SL}_2(q)\times SL_2(q)).\mathbb{Z}_{q-1}$		$\Phi_3\Phi_4$
$\hat{}(\mathbb{Z}_{q-1})^3.S_4$	$q\geq 5$	$q^6\Phi_2^2\Phi_3\Phi_4/24$
$\hat{}(\mathrm{SL}_2(q)\times\mathrm{SL}_2(q)).\mathbb{Z}_{q-1}.2$	$q\geq 4$	$q^4\Phi_3\Phi_4/2$
$(\mathrm{PSL}_2(q^2)\times\mathbb{Z}_{q+1}).2$		$q^4\Phi_1^2\Phi_3(q-1,2)/2d$
$\mathrm{PSL}_4(q_0).(q_0-1,4)$	$q=q_0^b$, b prime	$\frac{\lvert\mathrm{PSL}_4(q)\rvert}{(q_0-1,4)\lvert\mathrm{PSL}_4(q_0)\rvert}$
$2^4.S_6$	$q=p\equiv 1 \bmod 8$	$\lvert\mathrm{PSL}_4(q)\rvert/11520$
$2^4.A_6$	$q=p\equiv 5 \bmod 8$	$\lvert\mathrm{PSL}_4(q)\rvert/5760$
$\hat{}\mathrm{Sp}_4(q)\cdot(q-1,2)$		$q^2\Phi_1\Phi_3/(q-1,2)$
$\mathrm{PSO}_4^+(q).2$	q odd	$q^4\Phi_1\Phi_3\Phi_4/d$
$\mathrm{PSO}_4^-(q).2$	q odd	$q^4\Phi_1^2\Phi_2\Phi_3/d$
$\mathrm{PSU}_4(q_0).(q-1,2)$	$q=q_0^2$	$\frac{\lvert\mathrm{PSL}_4(q)\rvert}{(q-1,2)\lvert\mathrm{PSU}_4(q_0)\rvert}$
A_7	$q=p\equiv 1,2,4 \bmod 7$	$\lvert\mathrm{PSL}_4(q)\rvert/2520$
$\mathrm{PSU}_4(2)$	$q=p\equiv 1 \bmod 6$	$\lvert\mathrm{PSL}_4(q)\rvert/25920$

TABLE 4. Maximal subgroups of $\mathrm{PSL}_2(q)$ (see [16]).

Subgroup	Condition	Index
$D_{(q-1)}$	$q\geq 13$, odd	$\frac{1}{2}q\Phi_2$
$D_{2(q-1)}$	q even	$\frac{1}{2}q\Phi_2$
$D_{(q+1)}$	$q\neq 7,9$, odd	$\frac{1}{2}q\Phi_1$
$D_{2(q+1)}$	q even	$\frac{1}{2}q\Phi_1$
Borel subgroup		Φ_2
$\mathrm{PSL}_2(q_0)(2,\alpha)$	$q=q_0^\alpha$	
S_4	$q=p\equiv\pm1 \bmod 8$	
	$q=p^2, 3<p\equiv\pm3 \bmod 10$	
A_4	$q=p\equiv\pm3 \bmod 8, q>3$	
A_5	$q=p\equiv\pm1 \bmod 10$	
	$q=p^2, p\equiv\pm3 \bmod 10$	

(iii) $U/M\cong P_b\cong\tau(\hat{}[q^4]:(\mathrm{SL}_2(q)\times\mathrm{SL}_2(q)).\mathbb{Z}_{q-1})$, $|G':U|=\Phi_3\Phi_4$ *and for every* i, $t\phi_i(1)$ *divides some member of the set* $\mathcal{A}_2$, *where*

$$\mathcal{A}_2=\{q\Phi_1,\Phi_1^2,q\Phi_2,\Phi_2^2,\Phi_1\Phi_2,q^2\}.$$

Proof. Recall that, for every i, $t\phi_i|G:U|$ divides some degree of $\mathrm{PSL}_4(q)$. By a routine check on Tables 1 and 3, we see that one of the mentioned cases must happen. $\square$

TABLE 5. Maximal subgroups of $\mathrm{PSL}_3(q)$ (see [16]).

Subgroup	Condition
$\hat{}[q^2] : \mathrm{GL}_2(q)$	
$\hat{}(\mathbb{Z}_{q-1})^2.S_3$	$q \geq 5$
$\hat{}\mathbb{Z}_{q^2+q+1}.3$	$q \neq 4$
$\mathrm{PSL}_3(q_0).((q-1,3),b)$	$q = q_0^b$, b prime
$3^2.\mathrm{SL}_2(3)$	$q = p \equiv 1 \bmod 9$
$3^2.Q_8$	$q = p \equiv 4, 7 \bmod 9$
$\mathrm{SO}_3(q)$	q odd
$\mathrm{PSU}_3(q_0)$	$q = q_0^2$
A_6	$p \equiv 1, 2, 4, 7, 8, 13 \bmod 15$
$\mathrm{PSL}_2(7)$	$2 < q = p \equiv 1, 2, 4 \bmod 7$

TABLE 6. Maximal subgroups of $\mathrm{PSp}_4(q)$ (see [16]).

Subgroup	Condition
$\hat{}[q^3] : (\mathbb{Z}_{q-1} \circ \mathrm{Sp}_2(q))$	
$\hat{}[q^3] : \mathrm{GL}_2(q)$	
$(\mathrm{Sp}_2(q) \circ \mathrm{Sp}_2(q)).2$	$q \geq 3$
$\hat{}\mathrm{GU}_2(q).2$	q odd, $q \geq 5$
$\mathrm{PSp}_2(q^2).2$	
$\hat{}\mathrm{GL}_2(q).2$	q odd, $q \geq 5$
$\mathrm{PSp}_4(q_0).(b,(q-1,2))$	$q = q_0^b$, b prime
$2^4.\Omega_4^-(2)$	$q = p \equiv \pm 3 \bmod 8$
$2^4.\Omega_4^-(2)$	$q = p \equiv \pm 1 \bmod 8$
$\mathrm{O}_4^+(q)$	q even
$\mathrm{O}_4^-(q)$	q even
$\mathrm{Sz}(q)$	$q \geq 8$, q even, $\log_p(q)$ odd
$\mathrm{PSL}_2(q)$	$p \geq 5$, $q \geq 7$
S_6	$q = p \equiv \pm 1 \bmod 12$
A_6	$q = p \equiv 2, \pm 5 \bmod 12$

The next three lemmas can be seen directly from the list of maximal subgroups of the corresponding groups. We leave the detailed proofs to the reader.

Lemma 7.2. *If K is a maximal subgroup of $\mathrm{SL}_2(q)$ whose index divides Φ_1, Φ_2, or q, then K is the Borel subgroup of index Φ_2. Moreover, Φ_2 is the smallest index of a proper subgroup of* $\mathrm{SL}_2(\mathrm{q})$.

Lemma 7.3. *If K is a maximal subgroup of $\mathrm{SL}_3(q)$ whose index divides some member of the set $\mathcal{A}_1$ consisting of*

$$q\Phi_2, \Phi_1^2\Phi_2, \Phi_1\Phi_3, q\Phi_3, q^3, \Phi_2\Phi_3,$$

then $K \cong [q^2] : \mathrm{GL}_2(q)$ with index Φ_3.

Lemma 7.4. *The smallest index of a maximal subgroup of $\mathrm{Sp}_4(q)$ is Φ_4, which is greater than $2\Phi_1$.*

Lemma 7.5. *Let $N \trianglelefteq A$ be such that $A/N \cong \hat{S}$, where $\hat{S}/Z(\hat{S}) \cong \mathrm{PSL}_2(q)$ and let $\lambda \in \mathrm{Irr}(N)$. If $\chi(1) \leq \Phi_2$ for any $\chi \in \mathrm{Irr}(A|\lambda)$, then λ is A-invariant.*

Proof. By way of contradiction, assume that λ is not A-invariant. Let $J = I_A(\lambda)$ and $K \leq A$ such that $J \leq K \lneq A$, where K/N is a maximal subgroup of $A/N \cong \hat{S}$. Let $\delta \in \mathrm{Irr}(J|\lambda)$ and $\mu = \delta^A \in \mathrm{Irr}(A|\lambda)$. Then

$$\mu(1) = |A : K||K : J|\delta(1).$$

As $|A : K| \geq \Phi_2$ by Lemma 7.2 and that $\mu(1) \leq \Phi_2$, we deduce that $|K : J|\delta(1) = 1$. This implies that $K/N = J/N$ is isomorphic to the Borel subgroup of $\hat{S}$ of index Φ_2 and that all the irreducible constituents of λ^J are linear. By applying Gallagher's Lemma, we deduce that the Borel subgroup J/N is abelian, which is impossible as $q \geq 13$. □

Lemma 7.6. *Let $N \trianglelefteq A$ be such that $A/N \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and let $\lambda \in \mathrm{Irr}(N)$. If $\chi(1) \leq \Phi_2^2$ for any $\chi \in \mathrm{Irr}(A|\lambda)$, then λ is A-invariant.*

Proof. Let A_1 and A_2 be normal subgroups of A such that $A_i/N \cong \mathrm{SL}_2(q)$ for $i = 1, 2$. By way of contradiction, assume that λ is not A-invariant. As $A = A_1A_2$, λ is not A_i-invariant for some i. Without loss, we assume that λ is not A_1-invariant. Let $V = A_1$. Then λ is not V-invariant and $A/V \cong \mathrm{PSL}_2(q)$. Let $I = I_V(\lambda)$ and $K \leq A$ such that $I \leq K \lneq V$, where K/N is a maximal subgroup of $V/N \cong \mathrm{SL}_2(q)$. Let $\delta \in \mathrm{Irr}(I|\lambda)$ be such that $\delta(1)$ is maximal and let $\mu = \delta^V \in \mathrm{Irr}(V|\lambda)$. Then

$$\mu(1) = |V : K||K : I|\delta(1).$$

With the assumption on q, we obtain that $|V : K| \geq \Phi_2$ by Lemma 7.2 and thus

$$\mu(1) \geq |K : I|\Phi_2\delta(1).$$

If μ is not A-invariant, then with the same argument as above, we deduce that

$$\varphi(1) \geq \Phi_2\mu(1) \geq |K : I|\Phi_2^2\delta(1)$$

for any irreducible constituent φ of μ^A. As $\varphi \in \mathrm{Irr}(A|\lambda)$, we have that $\varphi(1) \leq \Phi_2^2$ and thus $|K : I|\delta(1) = 1$. It follows that $K/N = I/N$ is isomorphic to the Borel subgroup of $\mathrm{SL}_2(q)$ of index Φ_2 and every irreducible constituent of λ^I is linear. By applying Gallagher's Lemma, we deduce that I/N is abelian which is impossible as $q \geq 13$.

Therefore we conclude that μ is A-invariant. Write

$$\mu^A = \sum_{i=1}^{k} f_i\chi_i, \text{ where } \chi_i \in \mathrm{Irr}(A|\mu).$$

If $f_j = 1$ for some j, then μ extends to $\mu_0 \in \mathrm{Irr}(A)$ and so by applying Gallagher's Lemma, μ^A has an irreducible constituent of degree $\Phi_2\mu(1)$ since $\Phi_2 \in \mathrm{cd}(A/V)$. Using the same argument as in the previous case, we obtain a contradiction. Hence we have that all $f_i > 1$ and so $f_i's$ are degrees of nontrivial proper projective irreducible representations of $A/V \cong \mathrm{PSL}_2(q)$ by Lemma 2.7.

If $f_j = q+1$ for some j, then $\chi_j(1) = f_j\mu(1) = \Phi_2\mu(1)$, which leads to a contradiction as above.

Assume next that $f_j = q-1$ for some j. If $|K : I|\delta(1) \geq 2$, then as $q \geq 13$, we obtain

$$\chi_j(1) = (q-1)\mu(1) \geq (q^2-1)|K : I|\delta(1) \geq 2(q^2-1) > (q+1)^2 = \Phi_2^2,$$

which violates the hypothesis. Thus $|K : I|\delta(1) = 1$. It follows that $K = I$ and $\delta(1) = 1$. By the choice of δ, we deduce that all irreducible constituents of λ^I are linear and so $I/N = K/N$ is an abelian maximal subgroup of $V/N \cong \mathrm{SL}_2(q)$. It follows that $\mathrm{PSL}_2(q)$ has an abelian maximal subgroup, which is impossible since $q \geq 13$.

Thus for all i, we have that $f_i > 1$ and $f_i \neq q \pm 1$. By inspecting the character degrees of $\mathrm{SL}_2(q)$, we conclude that q is odd and $f_i = (q-\epsilon)/2$ for all i, where $q \equiv \epsilon \pmod 4$ and $\epsilon = \pm 1$. Note that when $q \geq 13$ is odd, the characters of degree q or $(q+\epsilon)/2$ of $\mathrm{SL}_2(q)$ are unfaithful so that these degrees are not the degrees of the proper projective irreducible representations of $\mathrm{PSL}_2(q)$. Now

$$\frac{1}{2}q(q^2-1) = |\mathrm{PSL}_2(q)| = |A/V| = \sum_{i=1}^{k} f_i^2 = k(\frac{q-\epsilon}{2})^2.$$

We obtain that

$$k(q-\epsilon) = 2q(q+\epsilon).$$

Hence $q-\epsilon \mid 2(q+\epsilon) = 2(q-\epsilon)+4\epsilon$ and thus $q-\epsilon$ divides 4, which is impossible as $q-\epsilon \geq q-1 \geq 12$. □

Lemma 7.7. *Let $N \trianglelefteq A$ be such that $A/N \cong \mathrm{SL}_3(q)$ and let $\lambda \in \mathrm{Irr}(N)$. If $\chi(1)$ divides some number in*

$$\mathcal{A}_1 = \{q\Phi_2, \Phi_1^2\Phi_2, \Phi_1\Phi_3, q\Phi_3, q^3, \Phi_2\Phi_3\}$$

for any $\chi \in \mathrm{Irr}(A|\lambda)$, then λ is A-invariant.

Proof. By way of contradiction, assume that λ is not A-invariant and let $J = I_A(\lambda)$. Write

$$\lambda^J = \delta_1 + \delta_2 + \cdots, \text{ where } \delta_i \in \mathrm{Irr}(J|\lambda).$$

We have that for every i, $\delta_i^A \in \mathrm{Irr}(A|\lambda)$ and $\delta_i^A(1) = |A : J|\delta_i(1)$ divides some number in $\mathcal{A}_1$. Let K be a subgroup of A such that $J \leq K$ and that K/N is maximal in A/N. It follows that the index $|A : K|$ divides some number in $\mathcal{A}_1$. By Lemma 7.3, we deduce that $K/N \cong [q^2] : \mathrm{GL}_2(q)$ and $|A : K| = \Phi_3$ so that

$$|K : J|\delta_i(1) \text{ divides } \Phi_1, \Phi_2 \text{ or } q.$$

Let L and V be subgroups of K such that $V/N \cong \mathrm{SL}_2(q)$ and $L/N \cong [q^2]$. Also let $W = LV \trianglelefteq K$. Observe that if $\chi \in \mathrm{Irr}(W|\lambda)$, then $\chi(1)$ must divide Φ_1, Φ_2 or q. In particular, if $\chi \in \mathrm{Irr}(V|\lambda)$, then $\chi(1) \leq \Phi_2$ by using Frobenius reciprocity. By Lemma 7.5, we obtain that λ is V-invariant. As the Schur multiplier of $V/N \cong \mathrm{SL}_2(q)$ with $q \geq 13$ is trivial, we deduce from [14, Theorem 11.7] that λ extends to $\lambda_0 \in \mathrm{Irr}(V)$. By Gallagher's Lemma, $\tau\lambda_0 \in \mathrm{Irr}(V|\lambda)$ for any $\tau \in \mathrm{Irr}(V/N)$. Choose $\tau \in \mathrm{Irr}(V/N)$ with $\tau(1) = \Phi_2$ and let $\mu = \tau\lambda_0$. Then $\mu(1) = \tau(1)\lambda(1) = \Phi_2\lambda(1)$.

Observe that if $\varphi \in \mathrm{Irr}(W|\mu)$, then $\mu(1) \leq \varphi(1) \leq \Phi_2$ so that $\lambda(1) = 1$ and $\varphi_V = \mu$ and thus by [14, Lemma 12.17], we have that $\mathbf{V}(\mu) \trianglelefteq W$, where $\mathbf{V}(\mu) \trianglelefteq V$ is the vanishing-off subgroup of μ. As V is non-normal in W, we deduce that $\mathbf{V}(\mu) \lneqq V$ and thus as $\mu_N = \Phi_2\lambda$, μ is nonzero on N, and so $N \leq \mathbf{V}(\mu) \lneqq V$. Hence $\mathbf{V}(\mu)/N$ lies inside the center of V/N. In particular,

$$|V : \mathbf{V}(\mu)| \text{ is divisible by } q(q^2-1)/(2, q-1). \tag{7.7}$$

On the other hand, by [14, Lemma 2.29], as μ vanishes on $V - \mathbf{V}(\mu)$ we have $[\mu_{\mathbf{V}(\mu)}, \mu_{\mathbf{V}(\mu)}] = |V : \mathbf{V}(\mu)|$. Suppose that $\mu_{\mathbf{V}(\mu)} = e\sum_1^t \theta_i$ where the $\theta_i \in \mathrm{Irr}(\mathbf{V}(\mu))$ are distinct and of equal degree by Clifford theory. We then have

$$|V : \mathbf{V}(\mu)| = [\mu_{\mathbf{V}(\mu)}, \mu_{\mathbf{V}(\mu)}] = e^2t \text{ and } \mu(1) = et\theta_i(1)$$

for all $1 \leq i \leq t$. Therefore, $|V : \mathbf{V}(\mu)| \leq \mu(1)^2 = \Phi_2^2$, which violates (7.7). The proof is complete. $\square$

We are now ready to eliminate the three remaining possibilities singled out in Lemma 7.1.

(i) **Case** $\mathbf{U/M \cong P_a} \cong \tau(\hat{}[\mathbf{q^3}] : \mathrm{GL}_{\mathbf{3}}(\mathbf{q}))$. Then $|G' : U| = \Phi_2\Phi_4$ and for every i, $t\phi_i(1)$ divides some member of the set $\mathcal{A}_1$, where

$$\mathcal{A}_1 = \{q\Phi_2, \Phi_1^2\Phi_2, \Phi_1\Phi_3, q\Phi_3, q^3, \Phi_2\Phi_3\}.$$

Let L, V and W be subgroups of U such that

$$L/M \cong [q^3], V/M \cong \mathrm{SL}_3(q), \text{ and } W = LV \trianglelefteq U.$$

We obtain that if $\chi \in \mathrm{Irr}(W|\theta)$, then $\chi(1)$ divides some number in $\mathcal{A}_1$.

(a) Subcase $L \leq I \leq U$. We have $W/L \cong \mathrm{SL}_3(q)$. Write $\theta^L = \sum_{i=1}^k \lambda_i$, where $\lambda_i \in \mathrm{Irr}(L|\theta)$.

We first show that $L/\operatorname{Ker}\theta$ is abelian. It suffices to show that $L' \leq \operatorname{Ker}\lambda_i$ for all i, and hence

$$L' \leq \cap_{i=1}^{k} \operatorname{Ker}\lambda_i = \operatorname{Ker}\theta^L = \operatorname{Ker}\theta.$$

By way of contradiction, assume that $L/\operatorname{Ker}\theta$ is nonabelian. Then $L' \nleq \operatorname{Ker}\lambda_j$ for some j, hence $\lambda_j(1) > \theta(1) = 1$ and $p \mid \lambda_j(1)$. As $\operatorname{Irr}(W|\lambda_j) \subseteq \operatorname{Irr}(W|\theta)$, by Lemma 7.7, we obtain that λ_j is W-invariant.

As $q \geq 8$, we deduce that the Schur multiplier of $W/L \cong SL_3(q)$ is trivial so that by [14, Theorem 11.7], λ_j extends to $\lambda_0 \in \operatorname{Irr}(W)$ and hence by Gallgher's Lemma, $\lambda_0\tau$ are all the irreducible constituents of λ^W where $\tau \in \operatorname{Irr}(W/L)$. By choosing $\tau \in \operatorname{Irr}(W/L)$ with $\tau(1) = q^3$, we obtain that $\lambda_0(1)\tau(1) = q^3\lambda_j(1)$ must divide some number in $\mathcal{A}_1$, which is impossible as $\lambda_j(1) > 1$.

Thus $L/\operatorname{Ker}\theta$ is abelian. By Lemma 7.7 again, all λ_i are W-invariant and linear. Therefore

$$[W, L] \leq \cap_{i=1}^{k} \operatorname{Ker}\lambda_i \leq \operatorname{Ker}\theta^L = \operatorname{Ker}\theta \leq M,$$

which is a contradiction as W acts nontrivially on $L/M \cong [q^3]$.

(b) Subcase $L \nleq I$. As $I \cap L \lneq L$, we can choose $\lambda \in \operatorname{Irr}(L|\theta)$ with $p \mid \lambda(1)$. By Lemma 7.7, we obtain that λ is W-invariant. As $q \geq 13$, we deduce that the Schur multiplier of $W/L \cong \mathrm{SL}_3(q)$ is trivial so that by [14, Theorem 11.7], λ extends to $\lambda_0 \in \operatorname{Irr}(W)$ and hence by Gallgher's Lemma, $\lambda_0\tau$ are all the irreducible constituents of λ^W where $\tau \in \operatorname{Irr}(W/L)$. By choosing $\tau \in \operatorname{Irr}(W/L)$ with $\tau(1) = q^3$, we obtain that $\lambda_0(1)\tau(1) = q^3\lambda(1)$ must divide some number in $\mathcal{A}_1$, which is impossible as $\lambda(1) > 1$.

(ii) **Case** $\mathbf{U/M} \cong \tau({}^\wedge\mathrm{Sp}_{\mathbf{4}}(\mathbf{q}) \cdot (\mathbf{q-1, 2})) \cong \mathrm{S}_4(q) \cdot [(q-1,2)^2/(q-1,4)]$. Then $|G' : U| = q^2\Phi_1\Phi_3/(2, q-1)$, and for every i, $t\phi_i(1)$ divides $(2, q-1)\Phi_1$. Let $M \trianglelefteq W \trianglelefteq U$ be such that $W/M \cong \mathrm{S}_4(q)$. As $W \trianglelefteq U$, for any $\varphi \in \operatorname{Irr}(W|\theta)$, we obtain that $\varphi(1)$ divides $(2, q-1)\Phi_1$. By Lemma 7.4, we deduce that θ is W-invariant. Write $\theta^W = \sum_{i=1}^{k} f_i\mu_i$, where $\mu_i \in \operatorname{Irr}(W|\theta)$. If $f_j = 1$ for some j, then $\theta \in \operatorname{Irr}(M)$ is extendible to $\theta_0 \in \operatorname{Irr}(W)$. By Lemma 2.2 we obtain that $\tau\theta_0$ are all the irreducible constituents of θ^W for $\tau \in \operatorname{Irr}(W/M)$. As $W/M \cong \mathrm{S}_4(q)$, it has an irreducible character $\tau \in \operatorname{Irr}(W/M)$ with $\tau(1) = q^4$, and hence $\theta_0(1)\tau(1) = q^4$ must divide $(2, q-1)\Phi_1$, which is impossible. Hence this case cannot happen. Thus all $f_i > 1$ and they are the degree of projective irreducible representation of $W/M \cong \mathrm{S}_4(q)$. As $W \trianglelefteq U$, we deduce that $\varphi(1)$ divides $(2, q-1)\Phi_1$ for any $\varphi \in \operatorname{Irr}(W|\theta)$. Therefore as $\mu_i(1) = f_i\theta(1) = f_i$, we deduce that $f_i(1)$ divide $(2, q-1)\Phi_1$ for all i. In particular, we have $f_i \leq 2\Phi_1$ for all i. However using [18], we see that the smallest nontrivial projective degree of $S_4(q)$ when $q > 3$ is $\frac{1}{2}\Phi_1\Phi_2$ which is obviously larger then $2\Phi_1$ as $q \geq 13$. Thus this case cannot happen.

(iii) **Case** $\mathbf{U/M} \cong \mathbf{P_b} \cong \tau({}^\wedge[\mathbf{q^4}] : (\mathrm{SL}_{\mathbf{2}}(\mathbf{q}) \times \mathrm{SL}_{\mathbf{2}}(\mathbf{q})).\mathbb{Z}_{\mathbf{q-1}}))$. Then $|G' : U| = \Phi_3\Phi_4$ and for every i, $t\phi_i(1)$ divides some member of the set

$$\mathcal{A}_2 = \{q\Phi_1, \Phi_1^2, q\Phi_2, \Phi_2^2, \Phi_1\Phi_2, q^2\}.$$

Let L, V_1, V_2 be subgroups of U containing M such that

$$L/M \cong [q^4], V_i/M \cong \mathrm{SL}_2(q) \text{ for } i = 1, 2,$$

and let

$$V = V_1V_2, W = LV.$$

Then $W \trianglelefteq U$ and for any $\varphi \in \mathrm{Irr}(W|\theta)$, we obtain that $\varphi(1)$ divides some number in $\mathcal{A}_2$. In particular, $\varphi(1) \le \Phi_2^2$. By Frobenius reciprocity and the fact that $(\theta^V)^W = \theta^W$, we obtain that $\chi(1) \le \Phi_2^2$ for any $\chi \in \mathrm{Irr}(V|\theta)$. By Lemma 7.6, we have that θ is V-invariant and hence $V \le I$. Let

$$L_1 = L \cap I \text{ and } X = W \cap I.$$

We have that $L_1 \trianglelefteq X$ and $W = LX$ since $W = LV$ and $V \le I$.

(a) Subcase $L \le I$. Then θ is W-invariant. We have $W/L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and W acts irreducibly on $L/M \cong [q^4]$. Write

$$\theta^L = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \text{ where } \lambda_i \in \mathrm{Irr}(L|\theta).$$

We will show that $L/ker\theta$ is abelian. It suffices to show that $L' \le \mathrm{Ker}\,\lambda_i$ for all i, and hence

$$L' \le \cap_{i=1}^k \mathrm{Ker}\,\lambda_i = \mathrm{Ker}\,\theta^L = \mathrm{Ker}\,\theta.$$

By way of contradiction, assume that $L/\mathrm{Ker}\,\theta$ is nonabelian. Then $L' \nleq \mathrm{Ker}\,\lambda_j$ for some j, hence $\lambda_j(1) > \theta(1) = 1$ and $p \mid \lambda_j(1)$. By Lemma 7.6, we deduce that λ_j is W-invariant. By [1, Theorem 2], we deduce that L/M is an abelian chief factor of W, and so by Lemma 2.9, we obtain that $\lambda_j(1) = q^2$. Hence for any $\delta \in \mathrm{Irr}(W|\lambda_j)$, we have that $q^2 \mid \delta(1)$ and $\delta(1)$ divides one of the number in $\mathcal{A}_2$, so that $\delta(1) = q^2$. By [22, Theorem 2.3], we obtain a contradiction as $W/L \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ is nonsolvable.

Thus $L/\mathrm{Ker}\,\theta$ is abelian. Hence $\lambda_i(1) = 1$ and also λ_i are W-invariant for all i by Lemma 7.6. Hence we obtain that

$$[W, L] \le \cap_{i=1}^k \mathrm{Ker}\,\lambda_i \le \mathrm{Ker}\,\theta^L = \mathrm{Ker}\,\theta \le M,$$

which is a contradiction as W acts nontrivially on $L/M \cong [q^4]$.

(b) Subcase $L \nleq I$. Then $L_1/M \lneq L/M$. We have that $L_1/M \trianglelefteq X/M$ and since $L/M \cong [q^4]$ is abelian, we also have $L_1/M \trianglelefteq L/M$, and hence

$$L_1/M \trianglelefteq XL/M = W/M.$$

As $L_1/M \lneq L/M$ and L/M is irreducible under W/M, we deduce that $L_1 = M$. Then

$$t = |U : WI| \cdot |W : X| = |U : WI| \cdot |L : L_1| = |U : WI| \cdot |L : M| = q^4|U : WI|.$$

Hence $q^4 \mid t$, which is impossible as t divides some number in $\mathcal{A}_2$.

8. Final step

Recall from section 6 that

$$G/M \cong G'/M \times C/M,$$

where $C/M = C_{G/M}(G'/M)$ and $G'/M \cong \mathrm{PSL}_4(q)$.

We first claim that C/M is abelian. Assume not, then it would have a nonlinear irreducible character, say ψ. It follows that $\psi \times \mathrm{St}_{G'/M}$ is an irreducible character of G/M of degree $q^6\psi(1)$, which is impossible.

(i) We assume that $M = M'$. If M is abelian then $M = M' = 1$. It follows that $G \cong G' \times C_G(G')$ and we are done as $C_G(G') \cong G/G'$ is abelian. So it remains to consider the case when M is non-abelian. Let $N \leq M$ be a normal subgroup of G' so that $M/N \cong S^k$ for some non-abelian simple group S. By [21, Lemma 4.2], S has a non-principal irreducible character φ extending to $\mathrm{Aut}(S)$. Lemma 2.6 and its proof then imply that φ^k extends to G'/N. Therefore, by Gallagher's lemma, $\varphi^k\mathrm{St}_{G'/M} \in \mathrm{Irr}(G'/N)$ where $\mathrm{St}_{G'/M}$ is the Steinberg character of $G'/M \cong \mathrm{PSL}_4(q)$. However, $\varphi^k(1)\mathrm{St}_{G'/M}(1)$, which is $q^6\varphi^k(1)$, does not divide any degree of G and hence we get a contradiction.

(ii) Now we can assume $M > M'$. As shown in section 7, every linear character of M is G'-invariant. Using Lemma 2.8, we deduce that

$$[M, G'] = M' \text{ and } |M : M'| \mid |\mathrm{Mult}(G'/M)|.$$

Remark that $\mathrm{Mult}(\mathrm{PSL}_4(q))$ is a cyclic group of order $(q-1, 4)$. We see that q must be odd and we come up with the two following cases:

Case $\mathbf{q \equiv 3}$ mod 4: Then $|\mathrm{Mult}(G'/M)| = 2$ and hence $|M : M'| = 2$. Therefore, by Lemma 2.10, we have that $G'/M' \cong \mathrm{SL}_4(q)$. For any $x, y \in G'$ and $c \in C$, the fact $[M, G'] = M'$ implies that

$$[xM', yM']^{cM'} = [xM'^{cM'}, yM'^{cM'}] = [xM'mM', yM'nM'] = [xM', yM']$$

for some $m, n \in M$. In other words, C centralizes G'/M' and we obtain that G/M' is an internal central product of G'/M' and C/M' with amalgamated $M/M' \cong \mathbb{Z}_2$.

We claim that if C/M' is abelian then $\mathrm{cd}(G/M') = \mathrm{cd}(G'/M')$. The inclusion $\mathrm{cd}(G/M') \subseteq \mathrm{cd}(G'/M')$ is obvious. Thus it remains to show that $\mathrm{cd}(G'/M') \subseteq \mathrm{cd}(G/M')$. Let tM' be the generator of M/M' and $\chi \in \mathrm{Irr}(G'/M')$. As tM' is a central involution of G'/M', we have $\chi(tM') = \chi(1)$ or $-\chi(1)$. Since tM' is also a central involution of the abelian group C/M', there exists a linear character $\theta \in \mathrm{Irr}(C/M')$ such that $\theta(tM') = \chi(1)/\chi(tM')$. We then have

$$\chi \times \theta \in \mathrm{Irr}(G'/M' \times C/M')$$

and furthermore

$$(tM', tM') \in \mathrm{Ker}(\chi \times \theta).$$

It follows that $\chi \times \theta$ can be considered as a character of G/M' and hence we have proved that $\chi(1) \in \mathrm{cd}(G/M')$, as wanted.

From the above claim, if C/M' is abelian, we would have

$$\mathrm{cd}(SL_4(q)) = \mathrm{cd}(G'/M') = \mathrm{cd}(G/M') \subseteq \mathrm{cd}(G) = \mathrm{cd}(\mathrm{PSL}_4(q)).$$

This however contradicts Lemma 3.5 that $\Phi_1\Phi_2\Phi_3\Phi_4/2$ is in $\mathrm{cd}(\mathrm{SL}_4(q))$ but not in $\mathrm{cd}(\mathrm{PSL}_4(q))$. Thus we obtain that C/M' is non-abelian. Let α be a nonlinear irreducible character of C/M'. As C/M is abelian, α must be faithful and hence $\alpha(tM') = -\alpha(1)$. On the other hand, by inspecting the character degrees and their multiplicities of $\mathrm{SL}_4(q)$ and $\mathrm{PGL}_4(q)$ from [18], one sees that both have degree $\Phi_1^2\Phi_2^2\Phi_4$ with multiplicity $q(q^2-1)/3$. This in particular implies that $G'/M' \cong \mathrm{SL}_4(q)$ has a faithful irreducible character of degree $\Phi_1^2\Phi_2^2\Phi_4$, say β. We then have

$$\beta(tM') = -\beta(1) = -\Phi_1^2\Phi_2^2\Phi_4.$$

Therefore,

$$\alpha(tM')\beta(tM') = \alpha(1)\beta(1).$$

In other words, $(tM', tM') \in \mathrm{Ker}(\beta \times \alpha)$. As G/M' is an internal central product of G'/M' and C/M' with amalgamated M/M', we obtain

$$\beta \times \alpha \in \mathrm{Irr}(G/M').$$

This however is a contradiction since $\alpha(1)\beta(1) = \Phi_1^2\Phi_2^2\Phi_4\alpha(1) > \Phi_1^2\Phi_2^2\Phi_4$ and G has no degree which is a proper multiple of $\Phi_1^2\Phi_2^2\Phi_4$.

Case $\mathbf{q \equiv 1}$ mod $\mathbf{4}$: Then $|\mathrm{Mult}(G'/M)| = 4$ and hence $|M : M'| = 2$ or 4. The case $|M : M'| = 2$ can be handled similarly as above with the notice that

$$G'/M' \cong \mathrm{SL}_4(q)/\langle -I\rangle,$$

where I is the identity 4×4 matrix. So we assume from now on that $|M : M'| = 4$. It follows by Lemma 2.10 that $G'/M' \cong \mathrm{SL}_4(q)$. Arguing as in the case $q \equiv 3 \bmod 4$, we obtain that G/M' is an internal central product of G'/M' and C/M' with amalgamated $M/M' \cong \mathbb{Z}_4$. If C/M' is abelian, as above, we would have $\mathrm{cd}(SL_4(q)) \subseteq \mathrm{cd}(\mathrm{PSL}_4(q))$. This again is impossible by Lemma 3.5. Therefore we have that C/M' is non-abelian. Let α be a nonlinear irreducible character of C/M'. Then $\alpha(tM') = \kappa\alpha(1)$ where κ is a nontrivial fourth root of 1 and tM' is the generator of M/M'. Inspecting the character degrees and their multiplicities of $\mathrm{SL}_4(q)$ and $\mathrm{PGL}_4(q)$ from [18], one sees that both have degree $\Phi_1^2\Phi_2^2\Phi_4$ with multiplicity $q(q^2-1)/3$. Also, $\mathrm{SL}_4(q)$ has no degree $\Phi_1^2\Phi_2^2\Phi_4/2$ or $\Phi_1^2\Phi_2^2\Phi_4/4$. This in particular implies that $G'/M' \cong \mathrm{SL}_4(q)$ has a faithful irreducible character of degree $\Phi_1^2\Phi_2^2\Phi_4$, say β, such that

$$\beta(tM') = \kappa^3\beta(1) = \kappa^3\Phi_1^2\Phi_2^2\Phi_4.$$

Therefore,

$$\alpha(tM')\beta(tM') = \kappa^4\alpha(1)\beta(1) = \alpha(1)\beta(1).$$

In other words, $tM' \in \mathrm{Ker}(\alpha\beta)$ and hence

$$\alpha\beta \in \mathrm{Irr}(G'/M' \circ C/M') = \mathrm{Irr}(G/M').$$

This is a contradiction as $(\alpha\beta)(1) = \Phi_1^2\Phi_2^2\Phi_4\alpha(1) > \Phi_1^2\Phi_2^2\Phi_4$ and G has no degree which is a proper multiple of $\Phi_1^2\Phi_2^2\Phi_4$.

References

[1] H. Azad, M. Barry and G.M. Seitz. On the structure of parabolic subgroups. *Comm. Algebra* **18** (1990), 551 – 562.

[2] M. Bianchi, D. Chillag, M.L. Lewis and E. Pacifici. Character degree graphs that are complete graphs. *Proc. Amer. Math. Soc.* **135** (2007), 671 – 676.

[3] R.W. Carter. *Finite groups of Lie type: Conjugacy classes and complex characters.* (Wiley and Sons, New York, 1985).

[4] M. Claßen-Houben. *Die unipotenten Charaktere fur die GAP-Charaktertafeln der endlichen Gruppen vom Lie-Typ*, Diplomarbeit. RWTH Aachen (2005).

[5] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson. *Atlas of Finite Groups* (Clarendon Press, Oxford, 1985).

[6] E.C. Dade. Deux groupes finis distincts ayant la même algèbre de groupe sur tout corps. *Math. Z.* **119** (1971), 345 – 348.

[7] S. Dolfi, G. Navarro, and P.H. Tiep, Primes dividing the degrees of the real characters, *Math. Z.* **259** (2008), 755 – 774.

[8] P. Erdös and J. L. Selfridge. The product of consecutive integers is never a power. *Illinois J. Math* **19** (1975), 292 – 301.

[9] W. Feit. Extending Steinberg characters. *Contemp. Math.* **153** (1993), 1 – 9.

[10] D. Gorenstein, R. Lyons and R. Solomon. *The classifications of the finite simple groups.* (Math. Surveys Monogr., No. 3, Amer. Math. Soc., Providence, 1998).

[11] B. Huppert. Some simple groups which are determined by the set of their character degrees I. *Illinois J. Math.* **44** (2000), 828 – 842.

[12] B. Huppert. Some simple groups which are determined by the set of their character degrees. II. *Rend. Sem. Mat. Univ. Padova* **115** (2006), 1-13.

[13] B. Huppert. Some simple groups which are determined by the set of their character degrees IX. preprint.

[14] I.M. Isaacs. *Character Theory of Finite Groups*, (Dover Publications, Inc., New York, 1994).

[15] O. King. The subgroup structure of finite classical groups in terms of geometric configurations, *London Math. Soc. Lecture Note Series* **327** (2005), 29 – 56.

[16] P.B. Kleidman. *The Subgroup Structure of some Finite Simple Groups.* PhD thesis. Trinity College (1986).

[17] P. Kleidman and M. Liebeck. *The subgroup structure of the finite classical groups.* London Mathematical Society Lecture Note Series **129**, (Cambridge University Press, Cambridge, 1990).

[18] F. Lübeck. Data for finite groups of Lie type and related algebraic groups. `http://www.math.rwth-aachen.de/ Frank.Luebeck/chev/index.html`

[19] G. Malle and A.E. Zalesskii. Prime power degree representations of quasi-simple groups. *Arch. Math.* **77** (2001), 461 – 468.

[20] J. Moori and H.P. Tong-Viet. Character degree sets of $\Omega_7(q)$ and a conjecture of Huppert. Submitted (2011).

[21] A. Moretó. Complex group algebras of finite groups: Brauer's problem 1. *Adv. Math.* **208** (2007), 236 – 248.

[22] A. Moretó. An answer to a question of Isaacs on character degree graphs. *Adv. Math.* **201** (2006), $90-101$.
[23] T. Nagell. Des équations indéterminées $x^2+x+1=y^n$ et $x^2+x+1=3y^n$. *Norsk. Mat. Foren. Skr. Serie I* **2** (1921), $12-14$.
[24] G. Navarro, P.H. Tiep and A. Turull. Brauer characters with cyclotomic field of values. *J. Pure Appl. Algebra* **212** (2008), $628-635$.
[25] H.P. Tong-Viet. The simple Ree groups ${}^2\mathrm{F}_4(\mathrm{q}^2)$ are determined by the set of their character degrees. *J. Algebra.* **339** (2011), 357–369.
[26] H.P. Tong-Viet and T.P. Wakefield. On the Huppert's Conjecture for ${}^3\mathrm{D}_4(\mathrm{q}), \mathrm{q}\geq 3$. *Algebr Represent Theor.* DOI 10.1007/s10468-011-9316-0 (2011).
[27] H.P. Tong-Viet and T.P. Wakefield. On the Huppert's Conjecture for $\mathrm{G}_2(\mathrm{q}), \mathrm{q}\geq 7$. Submitted (2011).
[28] T.P. Wakefield. Verifying Huppert's conjecture for $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q^2)$. *Comm. Algebra* **37** (2009), 2887–2906.
[29] T.P. Wakefield. Verifying Huppert's conjecture for ${}^2\mathrm{G}_2(\mathrm{q}^2)$. *Algebr Represent Theor.* **14** (2011), no. 14, 609–623.
[30] T.P. Wakefield. Verifying Huppert's conjecture for $\mathrm{PSp}_4(\mathrm{q})$ when $q>7$. *Algebr Represent Theor.* DOI: 10.1007/s10468-010-9246-2 (2011).
[31] T.P. Wakefield. Verifying Huppert's conjecture for the simple groups of Lie type of rank two. PhD thesis. Kent State University (2008).
[32] K. Zsigmondy. Zur Theorie der Potenzreste, *Monath. Math. Phys.* **3** (1892), $265-284$.

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